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The force distribution on a slender twisted particle in a Stokes flow

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The theory of Stokes flows about slender particles is considered for the case in which the particle cross-section is arbitrary and non-uniform along the axis of the particle, subject to certain smoothness assumptions. As distinct from previous works on slender particles in which the particle is represented approximately by a line distribution of Stokeslets, the representation adopted here is the exact one of a surface distribution of Stokeslets. This representation may then be used to recover the familiar one-dimensional integral equation for the equivalent line density of Stokeslets, together with estimates for the range of particle shapes for which the equation is valid. Within this range it is found that there is a class of particles for which the established perturbation scheme, which is used to obtain a solution to the integral equation, is singular. This class of particle shapes is illustrated by the example of a uniformly twisted particle whose pitch of twist is large enough compared with the cross-sectional dimension to ensure the validity of the equation, but small enough to make the usual method of solution singular. It is shown how the equation may be transformed so that an approximate solution can be found by means of a regular perturbation scheme. The results indicate that, for each of the axial and transverse components of motion, there is an equivalent particle of circular cross-section for which the total force and couple are the same as for the original particle. However, the radii of the equivalent cylinders are different for each component, the transverse component being affected by the twist while the axial component is not.

1. Introduction

Recently a number of papers concerning the motion of a rigid slender particle in Stokes flows have appeared. Most of these have dealt with particles of circular cross-section (see Cox 1970, 1971; Tillett 1970), but Batchelor (1970) has extended this work to include particles whose cross-section is not only arbitrary but may also vary 'slowly' in size and shape along its length. The basic assumption made by all these authors is that the effect of the particle upon the exterior flow is the same as if the particle were replaced by a distribution of Stokeslets along its straight axis. This leads to an 'outer' solution which is matched onto an 'inner' solution, which is derived by assuming that the flow near the surface of the particle is essentially the same as that close to a particle with a uniform cross-section. The outcome of such an analysis is a one-dimensional integral equation for the line density of Stokeslets, in terms of the velocity of the particle relative to the ambient flow field. Batchelor (1970) has shown, in principle, how to obtain approximate solutions to this equation by a perturbation expansion in the small parameter $\epsilon = -1/\ln \alpha$, where $\alpha(\ll 1)$ is a measure of the aspect ratio of the particle.

The work described in this paper falls into two parts. The first part is a rederivation of Batchelor's integral equation via an alternative, though related, method. We consider the particle to be represented by a distribution of Stokeslets over the actual surface, rather than along its axis. This is known (Ladyzhenskaya 1963) to provide an exact solution to the exterior Stokes problem. The basic assumption made is that the Stokeslet distribution has similar smoothness properties to those enjoyed by the surface itself. If this is accepted then the integral equation for the Stokeslet distribution may be systematically approximated until Batchelor's equation is achieved, together with an estimate of the error involved in terms of the surface geometry. In this work we choose to specify the error we are prepared to admit (this is usually determined by the practical limits imposed by the method of solution of the integral equation) and thereby place a constraint upon the surface geometries we are entitled to discuss.

Among these surface geometries is a class for which Batchelor's perturbation scheme is singular. In § 3 we illustrate this case by considering in detail a particular example for which the analysis is relatively simple. The example chosen is that of a uniformly twisted particle whose cross-sectional size and shape (but not orientation) are uniform along the axis, and whose pitch of twist, while being sufficiently large for Batchelor's integral equation to be valid, is small enough for a straightforward perturbation scheme to be singular. It is shown, however, that the equation can be manipulated into a form for which the perturbation scheme is regular.

It is expected that as the twist becomes tighter the total force and couple on the particle will tend towards that of a particle of circular cross-section. This is borne out by the theory presented here, though there are different equivalent radii for the axial and transverse components of motion. The latter is affected by the presence of twist while the former is not.

The intended application of this theory is in the field of suspensions of macromolecular particles in which Brownian motion will be an important effect. This means that the special case of alignment of the particle with the direction of the flow is of no particular importance, though it is crucial for cases of simple shear flow in which Brownian motion is considered to be virtually absent (see Cox 1971). We have therefore allowed ourselves to ignore the precise detail of the flow, the forces and indeed the shape of the particle in the neighbourhood of the particle ends.

2. The approximate equation for the force distribution

It is assumed that the particle is moving freely in a Newtonian solvent (of viscosity μ) and of infinite extent. In the absence of the particle the velocity field is taken to be a linear function of position; the presence of the particle will perturb this imposed field. With the neglect of inertia the perturbation velocity $\mathbf{u}(\mathbf{x})$ satisfies Stokes's equations, vanishes at infinity and, as the particle is rigid, is a linear function of position on the particle surface, the no-slip condition being assumed to hold at the interface. If we presume that the motion of the particle is known, i.e. that $\mathbf{u}$ is given on the particle boundary B, then we have a well-defined Stokes problem whose solution predicts the force distribution on B to first order in the Reynolds number. There is therefore no need to consider the associated Oseen problem. Ladyzhenskaya (1963) has shown that the velocity field for any external Stokes flow may be generated exactly by a distribution of Stokeslets on the surface B. If we let this distribution be denoted by $-\mathbf{F}$ ($\mathbf{F}$ is then the force density experienced by the particle), the relationship between $\mathbf{F}$ and the perturbation velocity field is, in view of the condition at infinity,

$$\mathbf{u}(\mathbf{x}) = -(8\pi\mu L)^{-1}\int_B \mathbf{G}(\mathbf{x};\mathbf{y}).\mathbf{F}(\mathbf{y})\,dB(\mathbf{y}), \tag{2.1}$$

where $2L$ is the total length of the particle and

$$\mathbf{G}(\mathbf{x};\mathbf{y}) = \frac{\mathbf{I}}{|\mathbf{y}-\mathbf{x}|} + \frac{(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})}{|\mathbf{y}-\mathbf{x}|^3}. \tag{2.2}$$

(Note that throughout this work we are using the dyadic notation, see, for instance Happel & Brenner (1965).) Here $\mathbf{x}$ and $\mathbf{y}$ are position vectors made dimensionless with respect to the length scale L. Ladyzhenskaya (1963) has also shown that (2.1) is valid in a continuous manner as $\mathbf{x}$ approaches and assumes a position on B. Therefore, for $\mathbf{x}$ on B, (2.1) is an exact integral equation for the Stokeslet distribution with the 'known' linear velocity field $\mathbf{u}(\mathbf{x})$ acting as a forcing term. The determination of exact solutions to (2.1) for other than simple shapes (e.g. the sphere and ellipsoid, for which solutions are already known) is probably out of the question. We shall therefore consider only long slender shapes, for which approximate solutions are obtainable.

Batchelor (1970), in his recent work on slender-body theory, derived a one-dimensional integral equation for the effective line density distribution of Stokeslets. This equation is applicable to particles which have a slowly varying change of size and shape along their axes. We propose to rederive his equation, using (2.1) as a starting point, and to derive estimates for the range of rates of variation that are admissible.

The particle is assumed to be straight in the sense that there is a straight axis within the particle. We take the orientation of the axis to be denoted by the unit vector $\mathbf{e}$ and its total length to be $2L$. A typical point on the axis, relative to the midpoint, is given by $\xi L\mathbf{e}$, where $-1 \leqslant \xi \leqslant +1$. For each ξ the intersection of the particle surface and the plane through ξ orthogonal to $\mathbf{e}$ defines a contour bounding the cross-section. The contour is described by the vector function $\alpha L\mathbf{R}(\xi,\phi)$, which depends on both ξ and the 'angular' parameter ϕ. The dimensionless parameter α, called the aspect ratio of the particle, is so chosen that the maximum perimeter size is $2\pi\alpha L$. Since the particle is assumed to be slender we take $\alpha \ll 1$. The configuration is shown in figure 1 for the particular example of a uniformly twisted particle.

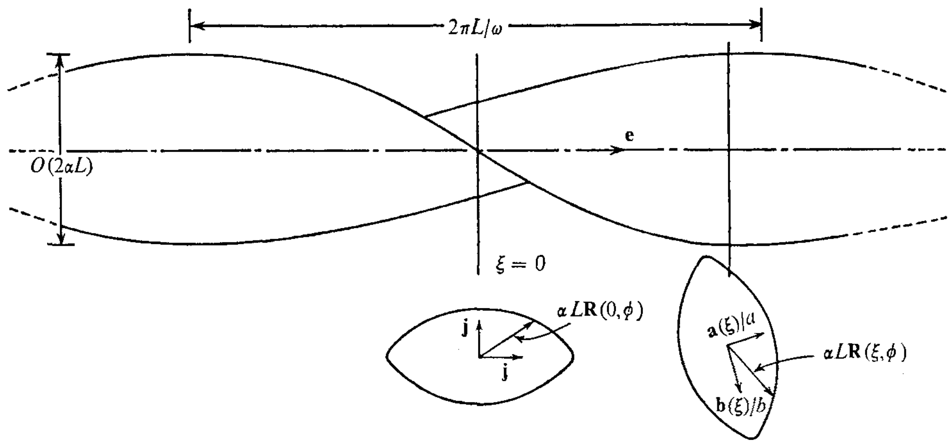

FIGURE 1. A portion of a uniformly twisted particle.

The surface B may be described by a vector function (made dimensionless with respect to L) of the two parameters ξ and ϕ in the form

$$\mathbf{x}(\xi,\phi) = \xi\mathbf{e}+\alpha\mathbf{R}(\xi,\phi). \tag{2.3}$$

Consequently, if we let the 'source point' in (2.2) be described by

$$\mathbf{y}(\eta,\theta) = \eta\mathbf{e}+\alpha\mathbf{R}(\eta,\theta), \tag{2.4}$$

we have $$\mathbf{y}-\mathbf{x} = (\eta-\xi)\,\mathbf{e}+\alpha\{\mathbf{R}(\eta,\theta)-\mathbf{R}(\xi,\phi)\}.$$

Let $$\mathbf{r} = \mathbf{R}(\eta,\theta)-\mathbf{R}(\xi,\phi), \quad r = |\mathbf{r}|, \tag{2.5}$$

then, after some rearrangement, we may write (2.2) in a form in which the smallness of α can be exploited. Thus

$$\mathbf{G}(\mathbf{x},\mathbf{y}) = \frac{\mathbf{I}+\mathbf{ee}}{\{(\eta-\xi)^2+\alpha^2r^2\}^{\frac{1}{2}}}+\frac{\alpha(\eta-\xi)\,(\mathbf{er}+\mathbf{re})+\alpha^2(\mathbf{rr}-r^2\mathbf{ee})}{\{(\eta-\xi)^2+\alpha^2r^2\}^{\frac{3}{2}}}. \tag{2.2a}$$

Here we have used the fact that $\mathbf{e}.\mathbf{r} = 0$. Also, from (2.4) we may calculate the surface element $dB(\mathbf{y})$:

$$dB(\mathbf{y}) = L^2\alpha\,d\eta\,d\theta\,|(\mathbf{e}+\alpha\mathbf{R}_\eta)\times\mathbf{R}_\theta|,$$

where the subscripts denote partial differentiation with respect to the variable indicated. From this we deduce that

$$dB(\mathbf{y}) = L^2\alpha\,|\mathbf{R}_\theta|\,d\eta\,d\theta\{1+O(\alpha^2|\mathbf{R}_\eta|^2)\}. \tag{2.6}$$

Consequently if we wish to use the approximate surface element $L^2\alpha\,d\eta\,d\theta\,|\mathbf{R}_\theta|$, then we are restricted to considering particles whose shape varies only slowly along the axis in the sense that $\alpha^2|\mathbf{R}_\eta|^2$ is acceptably smaller than unity. The question of just how small this has to be is considered in what follows. It is evident that close to the ends of the particle $\alpha^2|\mathbf{R}_\eta|^2$ may well cease to be negligible, and so we cannot expect the analysis to hold in those regions. A separate consideration is required if a detailed knowledge of the force distribution about the ends is desired. We shall not do this but instead we shall assume that, although our analysis will produce inaccurate results for the force density close to the ends, the contributions from these regions to the total force and couple (or any other integral force parameter) will be negligible to the order to which we are working.

Equation (2.1) may now be written as

$$\mathbf{u}(\xi,\phi) = -\frac{\alpha L}{8\pi\mu}\oint d\theta\int_{-1}^{+1} d\eta|\mathbf{R}_\theta|\,\mathbf{G}(\xi,\phi;\eta,\theta).\mathbf{F}(\eta,\theta)\{1+O(\alpha^2|\mathbf{R}_\eta|^2)\}. \quad (2.7)$$

In the appendix we show that after making certain plausible assumptions it is possible to obtain an approximation to (2.7). This approximate equation is

$$\begin{aligned}\mathbf{u}(\xi,\phi) = -\frac{\alpha L}{8\pi\mu}\Bigg[&\oint d\theta|\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta).\left((\mathbf{I}+\mathbf{ee})\left\{\frac{2}{\epsilon}+2\ln 2+\ln\frac{1-\xi^2}{\Gamma^2}\right\}\right.\\ &\left.+2\left\{\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}-\mathbf{ee}\right\}\right)+(\mathbf{I}+\mathbf{ee}).\oint d\theta\int_{-1}^{+1}d\eta\\ &\times\left\{\frac{|\mathbf{R}_\theta(\eta,\theta)|\,\mathbf{F}(\eta,\theta)-|\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta)}{|\eta-\xi|}\right\}\Bigg]\{1+O(\epsilon^n)\}, \quad (2.8)\end{aligned}$$

where
$$\mathbf{\Gamma} = \mathbf{R}(\xi,\theta)-\mathbf{R}(\xi,\phi) \quad (2.9)$$
and
$$\epsilon = -1/\ln\alpha, \quad (2.10)$$

the latter being considered small. The error term in (2.8) depends upon the particle shape. It is shown in the appendix that to restrict the error to be $O(\epsilon^n)$ the slope of the particle surface should everywhere (except at the ends) be no larger than $O(\epsilon^{2n})$. This is the sense in which the particle may have slowly varying axial changes in its cross-sectional characteristics.

Now $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\xi\mathbf{e}+\alpha\mathbf{R}(\xi,\phi)) = \mathbf{u}(\xi)\{1+O(\alpha)\}$ and this implies that the right-hand side of (2.8) is independent of ϕ to the order indicated. This means that the integrals

$$\mathbf{U}_A = \mathbf{e}\frac{1}{2\pi}\oint d\theta|\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{e}.\mathbf{F}(\xi,\theta)\ln\Gamma, \quad (2.11)$$

$$\mathbf{U}_T = \frac{1}{2\pi}\oint d\theta|\mathbf{R}_\theta(\xi,\theta)|\left\{\mathbf{I}^*\ln\Gamma-\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}\right\}.\mathbf{F}(\xi,\theta), \quad (2.12)$$

where $\mathbf{I}^* = \mathbf{I}-\mathbf{ee}$, are both independent of ϕ for all ξ. This observation allows a simplification of these forms. We first define the equivalent line Stokeslet density $\mathbf{f}(\xi)$ by

$$\mathbf{f}(\xi) = \oint d\theta|\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta). \quad (2.13)$$

Equation (2.12) may be recognized as an expression for the velocity of translation (as $\mathbf{U}_T$ is independent of ϕ) of an infinite cylinder in a direction perpendicular to its generators in an otherwise quiescent fluid, inertia being negligible. Unfortunately the problem for the general contour shape is as yet unsolved. It is, however, possible to show that $\mathbf{U}_T$ is a linear function of $\mathbf{f}$, and, as Batchelor (1970) has shown, the proportionality tensor is symmetric, from which we may conclude that for each contour there exists an elliptic contour which produces the same relationship between $\mathbf{U}_T$ and $\mathbf{f}$. In the absence of any definite results for the general contour, we shall consider it to be replaced by its equivalent ellipse and leave the relationship between the two shapes as an open question. We may therefore take

$$\mathbf{R}(\xi,\theta) = \mathbf{a}(\xi)\cos\theta + \mathbf{b}(\xi)\sin\theta, \tag{2.14}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the scaled principal axes (i.e. $a^* = a\alpha L$ and $b^* = b\alpha L$ are the actual sizes of the principal axes). By use of the standard methods for plane Stokes flows past elliptic sections it can be shown that

$$\mathbf{U}_T = -\frac{1}{2\pi}\left\{\mathbf{I}^*\ln\frac{2}{a+b} + \frac{a}{a+b}\frac{\mathbf{aa}}{a^2} + \frac{b}{a+b}\frac{\mathbf{bb}}{b^2}\right\}.\mathbf{f}. \tag{2.15}$$

Similarly, we recognize (2.11) as a problem arising in two-dimensional potential theory, in that $\mathbf{e}.\mathbf{F}$ may be thought of as a source density distributed in such a way that the potential $\mathbf{U}_A.\mathbf{e}$ assumes a constant value on the contour. This problem may be reduced for the general contour, the result being

$$\mathbf{U}_A = (1/2\pi)\ln|W'(\infty)|\,\mathbf{ee}.\mathbf{f}(\xi), \tag{2.16}$$

where W is the conformal mapping of the exterior of the unit circle onto the exterior of the contour. For the case of the above elliptic section we would have

$$|W'(\infty)| = \tfrac{1}{2}(a+b). \tag{2.17}$$

Equation (2.8) may therefore be reduced to the one-dimensional integral equation

$$\mathbf{u}(\xi) = -\frac{\alpha L}{4\pi\mu\epsilon}\left\{(\mathbf{I}+\mathbf{ee}).\mathbf{f}(\xi) + \epsilon\left[\mathbf{H}.\mathbf{f}(\xi) + \tfrac{1}{2}(\mathbf{I}+\mathbf{ee}).\int_{-1}^{+1}\frac{\mathbf{f}(\eta)-\mathbf{f}(\xi)}{|\eta-\xi|}\,d\eta\right]\right\}(1+O(\epsilon^n)), \tag{2.18}$$

where the tensor $\mathbf{H}$ is given by

$$\mathbf{H} = 2\left\{\ln\frac{2}{|W'(\infty)|} - \tfrac{1}{2} + \tfrac{1}{2}\ln(1-\xi^2)\right\}\mathbf{ee} + \left\{\ln\frac{4}{a+b} + \frac{a}{a+b} + \tfrac{1}{2}\ln(1-\xi^2)\right\}\frac{\mathbf{aa}}{a^2} + \left\{\ln\frac{4}{a+b} + \frac{b}{a+b} + \tfrac{1}{2}\ln(1-\xi^2)\right\}\frac{\mathbf{bb}}{b^2}. \tag{2.19}$$

This is the integral equation of Batchelor (1970, equations (7.1) and (7.2)), with the tensor $\mathbf{H}$ being here given explicitly in terms of the equivalent elliptic sections.

3. The case of a uniformly twisted particle

As an example of a particle for which the theory of § 2 is applicable we take a particle of elliptic section which is of constant shape and size, but whose orientation varies in a uniform manner with ξ (see figure 1). Such a particle surface may be defined mathematically by taking $\mathbf{a}(\xi)$ and $\mathbf{b}(\xi)$, as introduced in (2.14), to be given by

$$\left.\begin{aligned}\mathbf{a}(\xi) &= a(\mathbf{i}\cos\tfrac{1}{2}\omega\xi + \mathbf{j}\sin\tfrac{1}{2}\omega\xi),\\ \mathbf{b}(\xi) &= b(-\mathbf{i}\sin\tfrac{1}{2}\omega\xi + \mathbf{j}\cos\tfrac{1}{2}\omega\xi),\end{aligned}\right\} \tag{3.1}$$

where $a\mathbf{i}$ and $b\mathbf{j}$ are the principal axes at $\xi = 0$, and a and b are constants, subject of course to the constraint that the perimeter is to be $2\pi\alpha L$. This means that the pitch of twist p will be given by

$$p = 4\pi L/\omega. \tag{3.2}$$

It has been shown (see appendix) that the present theory is valid for

$$|\mathbf{R}_\xi| = O(\alpha^{-1}\epsilon^{2n});$$

in this example $|\mathbf{R}_\xi| = O(\omega)$ and so we find that ω may lie in the range zero to $O(\alpha^{-1}\epsilon^{2n})$.

The tensor $\mathbf{H}$ (equation (2.19) together with (2.17)) may be written for this example as

$$\mathbf{H} = 2\left\{\ln\frac{4}{a+b} - \tfrac{1}{2} + \tfrac{1}{2}\ln(1-\xi^2)\right\}\mathbf{ee} + \psi(\xi)\,\mathbf{I}^* + \kappa\boldsymbol{\Omega}(\omega\xi), \tag{3.3}$$

where

$$\left.\begin{aligned} \psi(\xi) &= \ln\frac{4}{a+b} + \tfrac{1}{2} + \tfrac{1}{2}\ln(1-\xi^2), \\ \kappa &= \tfrac{1}{2}(a-b)/(a+b), \\ \boldsymbol{\Omega}(\omega\xi) &= (\mathbf{ii}-\mathbf{jj})\cos\omega\xi + (\mathbf{ij}+\mathbf{ji})\sin\omega\xi. \end{aligned}\right\} \tag{3.4}$$

The form of $\mathbf{H}$ suggests that (2.18) should be split into its axial and transverse components. Let

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{e}u_A + \mathbf{I}^*.\mathbf{u}_T, \quad \mathbf{e}.\mathbf{u}_T = 0, \\ \mathbf{f} &= \mathbf{e}f_A + \mathbf{I}^*.\mathbf{f}_T, \quad \mathbf{e}.\mathbf{f}_T = 0, \end{aligned}\right\} \tag{3.5}$$

and further let

$$\Delta\{\mathbf{f}\} = \frac{1}{2}\int_{-1}^{1} \frac{(\mathbf{f}(\eta) - \mathbf{f}(\xi))}{|\eta-\xi|}\,d\eta. \tag{3.6}$$

The axial component of (2.18) is, to within $O(\epsilon^n)$,

$$f_A = -\frac{2\pi\mu\epsilon}{\alpha L}u_A - \epsilon\left[\left\{\ln\frac{4}{a+b} - \tfrac{1}{2} + \tfrac{1}{2}\ln(1-\xi^2)\right\}f_A + \Delta\{f_A\}\right]. \tag{3.7}$$

We notice that this equation is independent of ω, that is, the axial component of the force distribution is independent of the twist to this order of approximation. The results of Batchelor (1970) for a cylindrical particle in axial motion may therefore be used for this component directly.

The transverse component of (2.18) is, to within $O(\epsilon^n)$,

$$\mathbf{f}_T = -\frac{4\pi\mu\epsilon}{\alpha L}\mathbf{u}_T - \epsilon[(\psi\mathbf{I}^* + \kappa\boldsymbol{\Omega}).\mathbf{f}_T + \Delta\{\mathbf{f}_T\}]. \tag{3.8}$$

If $\omega = O(1)$ then Batchelor's (1970) results may again be used directly. However, the theory is valid for values of ω for which this is not the case, since the usual perturbation procedure for solving (3.8) is singular. We shall now concentrate on this case. Let

$$\omega = \alpha^{-\tau}, \tag{3.9}$$

where τ, which may be a function of α and ϵ, is to be restricted by

$$\epsilon \ll \tau \leqslant 1 - 2n\epsilon\ln(1/\epsilon). \tag{3.10}$$

The upper restriction ensures that we are in the region of validity and the lower restriction suffices for the usual perturbation procedure to be singular.

We shall now show how (3.8) may be reduced to an equation for which ordinary perturbation methods are regular. The operator Δ has the following properties: (i) $\Delta\{\text{constant}\} = 0$, (ii) $\Delta\{\xi\} = -\xi$, (iii) if $\mathbf{g}(\xi)$ is a function of ξ only (rather than of $\omega\xi$) and is smooth in $-1 < \xi < +1$, but may have logarithmic singularities at $\xi = \pm 1$, then

$$\Delta\{\mathbf{g}\,.\,\mathbf{\Omega}\} = -\{\ln\omega + \gamma + \tfrac{1}{2}\ln(1-\xi^2)\}\,\mathbf{g}\,.\,\mathbf{\Omega}(1+O(\epsilon^m)) \tag{3.11}$$

for any $m > 0$, where γ ($\simeq 0{\cdot}577$) is Euler's constant (it is the presence of the term $\ln\omega = \tau/\epsilon$ in this type of expression which renders the perturbation scheme singular).

The form of (3.8) and property (iii) of Δ, together with the fact that $\mathbf{\Omega}\,.\,\mathbf{\Omega} = \mathbf{I}^*$, suggest that the solution should be sought in the form

$$\mathbf{f}_T = \mathbf{f}_T^{(1)}(\xi) + \mathbf{\Omega}(\omega\xi)\,.\,\mathbf{f}_T^{(2)}(\xi). \tag{3.12}$$

This form may be thought of as the sum of an 'average' term $\mathbf{f}_T^{(1)}$ and a rapidly oscillating term $\mathbf{\Omega}\,.\mathbf{f}_T^{(2)}$. We substitute (3.12) into (3.8) and compare the terms which vary slowly with ξ, and those which vary rapidly (i.e. those terms containing $\mathbf{\Omega}$), to find

$$\mathbf{f}_T^{(1)} = -\frac{4\pi\mu\epsilon}{\alpha L}\mathbf{u}_T - \epsilon\{\psi\mathbf{f}_T^{(1)} + \kappa\mathbf{f}_T^{(2)} + \Delta\{\mathbf{f}_T^{(1)}\}\},$$

$$\{[1+\epsilon(\psi - \ln\omega - \gamma - \tfrac{1}{2}\ln(1-\xi^2))]\,\mathbf{f}_T^{(2)} + \epsilon\kappa\,\mathbf{f}_T^{(1)}\}\,.\,\mathbf{\Omega} = 0.$$

We now introduce a constant parameter ζ which conveys the effects of the twist:

$$\zeta = 1/\{1-\tau+\epsilon[\ln(4/(a+b)) + \tfrac{1}{2} - \gamma]\}, \tag{3.13}$$

whence we may write $\qquad \mathbf{f}_T^{(2)} = -\epsilon\kappa\zeta\mathbf{f}_T^{(1)}.$

This means that the force density is of the form

$$\mathbf{f}_T = (\mathbf{I}^* - \epsilon\kappa\zeta\mathbf{\Omega})\,.\,\mathbf{f}_T^{(1)}, \tag{3.14}$$

where $\mathbf{f}_T^{(1)}$ satisfies the integral equation

$$\mathbf{f}_T^{(1)} = -(4\pi\mu\epsilon/\alpha L)\,\mathbf{u}_T - \epsilon[(\psi - \epsilon\kappa^2\zeta)\,\mathbf{f}_T^{(1)} + \Delta\{\mathbf{f}_T^{(1)}\}]. \tag{3.15}$$

This equation is similar to that for a particle with a circular cross-section, so the results of Batchelor (1970) are applicable. Together with (3.14) these results give the transverse force distribution

$$\mathbf{f}_T = -\frac{4\pi\mu\epsilon}{\alpha L}[1-\tfrac{1}{2}\epsilon\ln(1-\xi^2)](\mathbf{I}^* - \epsilon\kappa\zeta\mathbf{\Omega})\,.\left\{\frac{\mathbf{u}_T^{(0)}}{1+\epsilon c} + \frac{\mathbf{u}_T^{(1)}\xi}{1+\epsilon(c-1)}\right\}(1+O(\epsilon^2)), \tag{3.16}$$

where $\qquad \mathbf{u}_T = \mathbf{u}_T^{(0)} + \mathbf{u}_T^{(1)}\xi$

and

$$c = \ln\frac{4}{a+b} + \tfrac{1}{2} - \epsilon\kappa^2\zeta. \tag{3.17}$$

Similarly, the axial component may be written as

$$\mathbf{f}_A = -\frac{2\pi\mu\epsilon}{\alpha L}[1-\tfrac{1}{2}\epsilon\ln(1-\xi^2)]\left\{\frac{\mathbf{u}_A^{(0)}}{1+\epsilon d}+\frac{\mathbf{u}_A^{(1)}\xi}{1+\epsilon(d-1)}\right\}(1+O(\epsilon^2)), \tag{3.18}$$

where
$$\mathbf{u}_A = \mathbf{u}_A^{(0)}+\mathbf{u}_A^{(1)}\xi,$$

and
$$d = \ln\frac{4}{a+b}-\tfrac{1}{2}. \tag{3.19}$$

The integral force parameters such as the total force and couple on the particle are obtained by integration of these distributions. The axial force $\mathscr{F}_A$ is given by

$$\mathscr{F}_A = -4\pi\mu\epsilon L\{1+\epsilon(1-\ln 2)\}/(1+\epsilon d)\,u_A^{(0)}(1+O(\epsilon^2)). \tag{3.20}$$

For the transverse components it is evident that if ω is sufficiently large then the rapidly varying terms depending upon $\mathbf{\Omega}$ will in effect be self-cancelling. For this to occur it is sufficient for the relative error of $O(\epsilon/\omega)$, incurred in neglecting these contributions, to be $O(\epsilon^2)$, that is $\tau \geqslant \epsilon\ln 1/\epsilon$. We shall therefore (for simplicity) restrict ourselves to the range

$$\epsilon\ln(1/\epsilon) \leqslant \tau \leqslant 1-4\epsilon\ln(1/\epsilon). \tag{3.21}$$

The total transverse force $\mathbf{F}_T$ and the couple $\mathbf{C}$ about the centre of the particle are given by

$$\mathbf{F}_T = -8\pi\mu\epsilon L\{[1+\epsilon(1-\ln 2)]/(1+\epsilon c)\}\,\mathbf{u}_T^{(0)}(1+O(\epsilon^2)); \tag{3.22}$$

$$\mathbf{C} = -\tfrac{8}{3}\pi\mu\epsilon L^2\{[1+\epsilon(\tfrac{4}{3}-\ln 2)]/(1+\epsilon(c-1))\}\,\mathbf{e}\times\mathbf{u}_T^{(1)}(1+O(\epsilon^2)). \tag{3.23}$$

Although the rapidly varying part of the force distribution makes no direct contribution to the total force, it does have a residual effect in that the parameter c depends upon the twist (equation (3.17)). At first sight it may appear that this dependence is negligible to the order to which we are working. On closer inspection, however, it can be seen that c may become as large as $O(1/\ln(1/\epsilon))$ within the range of validity, which would make the contributions from the twist larger than the indicated error term.

It is not difficult to see that if the particle is in axial motion then it experiences a force which is approximately the same as that experienced by a particle of the same length but with a circular cross-section of radius r_A, where r_A is given by

$$r_A = \tfrac{1}{2}(a^*+b^*)(1+O(\epsilon)). \tag{3.24}$$

Similarly, for the particle in transverse motion the total force and couple are the same as those experienced by a particle of circular cross-section of radius r_T, where

$$r_T = \tfrac{1}{2}(a^*+b^*)\exp\left\{\frac{1}{4}\left(\frac{a^*-b^*}{a^*+b^*}\right)^2\Big/\left[\ln\frac{p}{\pi(a^*+b^*)}+\tfrac{1}{2}-\gamma\right]\right\}(1+O(\epsilon)). \tag{3.25}$$

The methods we have described in this paper should be applicable to a wider class of particle shapes than has been dealt with here, provided that the 'slenderness' property is available for exploitation. For instance, for a particle which is a slender cylinder coiled into a helix with a 'helical' diameter D and pitch P, we would expect to be able to use the analogue of the analysis presented here provided that $D/P = O(\epsilon^{2n})$ (if the working were being carried to order ϵ^n), ϵ of course being based upon the cylinder aspect ratio.

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Appendix

This appendix contains the detailed analysis by which we attempt to justify the passage from equation (2.7) to equation (2.8).

As the tensor **G** has a singularity at **y** = **x**, we remove a small portion of the surface around this point for separate consideration. Having done this, the remainder of the surface falls naturally into three other regions. To define these regions we introduce the small artificial parameters χ_1 and χ_2, whose orders of magnitude are chosen to suit the needs of the analysis. The four regions are defined as follows:

(i) $B_1 \equiv \{\phi-\chi_2 \leqslant \theta \leqslant \phi+\chi_2;\ \xi-\chi_1 \leqslant \eta \leqslant \xi+\chi_1\}$,

(ii) $B_2 \equiv \{\text{all } \theta;\ -1 \leqslant \eta < \xi-\chi_1\}$,

(iii) $B_3 \equiv \{\text{all } \theta;\ \xi+\chi_1 < \eta \leqslant 1\}$,

(iv) $B_4 \equiv \{\text{all } \theta \text{ not in } (\phi-\chi_2, \phi+\chi_2);\ \xi-\chi_1 \leqslant \eta \leqslant \xi+\chi_1\}$.

We shall now make the plausible assumption that the variations in **F** mirror the variations in the particle geometry. To render this idea more precise we assume that the particle surface is differentiable and that

$$\mathbf{F}_\theta = O(|\mathbf{R}_\theta|\,\mathbf{F}) = O(\mathbf{F}), \quad \mathbf{F}_\xi = O(|\mathbf{R}_\xi|\,\mathbf{F}). \tag{A 1}$$

The contributions to the integral in (2.7) from each of the four regions will be considered in turn.

Contribution from B_1. If we take χ_2, $\chi_1|\mathbf{R}_\xi|$ and $\alpha|\mathbf{R}_\xi|$ all to be $o(1)$ then it is clear that the contribution from B_1 will be, to within an error of $o(1)$, the same as if B_1 were a plane rectangle of size $2\chi_1$ by $2\chi_2|\mathbf{R}_\theta| = 2\chi_2^*$. Therefore, in B_1, $\mathbf{F}(\eta,\theta) = \mathbf{F}(\xi,\phi)\{1+o(1)\}$ and $|\mathbf{R}_\theta(\eta,\theta)| = |\mathbf{R}_\phi(\xi,\phi)|\{1+o(1)\}$, and hence

$$\int_{B_1} \mathbf{G}.\mathbf{F}\,dB = 4\mathbf{F}(\xi,\phi).\int_0^{\chi_2^*} dq \int_0^{\chi_1} ds \left\{\frac{1}{(s^2+q^2)^{\frac{1}{2}}} + \frac{(\hat{\mathbf{i}}\hat{\mathbf{i}}s^2+\hat{\mathbf{j}}\hat{\mathbf{j}}q^2)}{(s^2+q^2)^{\frac{3}{2}}}\right\}\{1+o(1)\},$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the base vectors defining the approximating plane rectangle. If we let $\chi_1 = o(\chi_2^*)$, then $\ln\chi_2^* = o(\ln\chi_1)$ and the above integral may be evaluated to yield the estimate

$$\int_{B_1} \mathbf{G}.\mathbf{F}\,dB = \mathbf{F}(\xi,\phi)\,O(\chi_1\ln\chi_1). \tag{A 2}$$

Hence we may neglect all contributions from B_1 provided that

$$\chi_1 = o(\chi_2) = o(1), \quad \alpha|\mathbf{R}_\xi| = o(1), \quad \chi_1|\mathbf{R}_\xi| = o(1), \quad \chi_1\ln\chi_1 = O(\epsilon^n). \tag{A 3}$$

Contribution from B_2. By use of the approximate surface element we may write

$$\int_{B_2} \mathbf{G}.\mathbf{F}\,dB = \oint d\theta \int_{-1}^{\xi-\chi_1} d\eta\,|\mathbf{R}_\theta(\eta,\theta)|\,\mathbf{G}.\mathbf{F}$$

$$= \oint d\theta \left\{ |\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta).\int_{-1}^{\xi-\chi_1} d\eta\,\mathbf{G} + \int_{-1}^{\xi-\chi_1} d\eta\,\mathbf{G}.\mathbf{N}(\eta,\xi,\theta)\right\},$$

where $$\mathbf{N}(\eta,\xi,\theta) = |\mathbf{R}_\theta(\eta,\theta)|\,\mathbf{F}(\eta,\theta) - |\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta).$$

As $\mathbf{r} = O(1)$ in this region, it is not difficult to show that

$$\int_{-1}^{\xi-\chi_1} d\eta\,\mathbf{G} = (\mathbf{I}+\mathbf{ee})\ln(1+\xi)/\chi_1 + O(\alpha/\chi_1). \tag{A 4}$$

Similarly $$\int_{-1}^{\xi-\chi_1} d\eta\,\mathbf{G}.\mathbf{N} = (\mathbf{I}+\mathbf{ee}).\int_{-1}^{\xi-\chi_1} d\eta \frac{\mathbf{N}}{|\eta-\xi|}(1+O(\alpha/\chi_1)).$$

Consequently, provided that

$$\alpha/\chi_1 = O(\epsilon^n), \tag{A 5}$$

$$\int_{B_2} \mathbf{G}.\mathbf{F}\,dB = (\mathbf{I}+\mathbf{ee}).\oint d\theta \left\{ |\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta)\frac{\ln(1+\xi)}{\chi_1} + \int_{-1}^{\xi-\chi_1} d\eta \frac{\mathbf{N}}{|\eta-\xi|}\right\}(1+O(\epsilon^n)). \tag{A 6}$$

Contribution from B_3. The arguments used to evaluate the contribution from B_2 are immediately applicable here. If (A 5) holds then

$$\int_{B_3} \mathbf{G}.\mathbf{F}\,dB = (\mathbf{I}+\mathbf{ee}).\oint d\theta \left\{ |\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta)\frac{\ln(1-\xi)}{\chi_1} + \int_{\xi+\chi_1}^{1} d\eta \frac{\mathbf{N}}{|\eta-\xi|}\right\}(1+O(\epsilon^n)). \tag{A 7}$$

Contribution from B_4. We write

$$\int_{B_4} \mathbf{G}.\mathbf{F}\,dB = \oint' d\theta \int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,|\mathbf{R}_\theta(\eta,\theta)|\,\mathbf{G}.\mathbf{F}$$

$$= \oint' d\theta \left\{ |\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta)\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G} + \int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G}.\mathbf{N}\right\},$$

where $\oint'$ denotes integration over all θ not in $(\phi-\chi_2, \phi+\chi_2)$. Consider $\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G}$ and let $\eta = \xi + \alpha z$, then

$$\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G} = \int_{\xi-\chi_1}^{\xi+\chi_1} dz \left\{ \frac{\mathbf{I}+\mathbf{ee}}{(z^2+r^2)^{\frac{1}{2}}} + \frac{z(\mathbf{er}+\mathbf{re})+(\mathbf{rr}-r^2\mathbf{ee})}{(z^2+r^2)^{\frac{3}{2}}}\right\},$$

and $$\mathbf{r} = \mathbf{R}(\xi+\alpha z,\theta)-\mathbf{R}(\xi,\phi) = \mathbf{\Gamma}+O(|\mathbf{R}_\xi|\,\chi_1),$$
with $\mathbf{\Gamma}$ as defined in (2.9). Therefore

$$\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G} = \int_{-\chi_1/\alpha}^{\chi_1/\alpha} dz\left\{\frac{\mathbf{I}+\mathbf{ee}}{(z^2+\Gamma^2)^{\frac{1}{2}}}+\frac{z(\mathbf{e\Gamma}+\mathbf{\Gamma e})+(\mathbf{\Gamma\Gamma}-\Gamma^2\mathbf{ee})}{(z^2+\Gamma^2)^{\frac{3}{2}}}\right\}(1+O(|\mathbf{R}_\xi|\,\chi_1))$$
$$= \left\{2(\mathbf{I}+\mathbf{ee})\left[\ln\left[\frac{\chi_1}{\alpha}+\left(\frac{\chi_1^2}{\alpha^2}+\Gamma^2\right)^{\frac{1}{2}}\right]-\ln\Gamma\right]\right.$$
$$\left.+2\left(\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}-\mathbf{ee}\right)\frac{\chi_1/\alpha}{(\chi_1^2/\alpha^2+\Gamma^2)^{\frac{1}{2}}}\right\}(1+O(|\mathbf{R}_\xi|\,\chi_1)).$$

Hence, by use of (A 5) and the condition (cf. (A 3))

$$|\mathbf{R}_\xi|\,\chi_1 = O(\epsilon^n), \tag{A 8}$$

we have that

$$\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G} = 2\left\{(\mathbf{I}+\mathbf{ee})\ln\frac{2\chi_1}{\alpha\Gamma}+\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}-\mathbf{ee}\right\}+O(\epsilon^n).$$

A similar argument leads to the result

$$\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\mathbf{G}.\mathbf{N} = O(\alpha|\mathbf{R}_\xi|\ln\alpha\mathbf{F}),$$

where use has been made of the assumption in (A 1). Consequently if we take

$$\alpha\,|\mathbf{R}_\xi|\ln\alpha = O(\epsilon^n) \tag{A 9}$$

then

$$\int_{B_4}\mathbf{G}.\mathbf{F}\,dB = 2\oint' d\theta\,|\mathbf{R}_\theta(\xi,\theta)|\,\mathbf{F}(\xi,\theta).\left\{(\mathbf{I}+\mathbf{ee})\ln\frac{2\chi_1}{\alpha\Gamma}+\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}-\mathbf{ee}\right\}(1+O(\epsilon^n)).$$

The integral $\oint'$ over this integrand may be replaced by the integral over all θ with an error of $O(\chi_2\ln\chi_2\mathbf{F})$, and so we further take

$$\chi_2\ln\chi_2 = O(\epsilon^n). \tag{A 10}$$

Finally, we note that

$$\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\frac{\mathbf{N}}{|\eta-\xi|} = O(|\mathbf{R}_\xi|\,\alpha\mathbf{F}), \tag{A 11}$$

which in conjunction with (A 8) allows us to write

$$\int_{B_4}\mathbf{G}.\mathbf{F}\,dB = \oint d\theta\left\{2\,|\mathbf{R}_\theta(\xi,\theta)|\mathbf{F}(\xi,\theta).\left[(\mathbf{I}+\mathbf{ee})\ln\frac{2\chi_1}{\alpha\Gamma}+\frac{\mathbf{\Gamma\Gamma}}{\Gamma^2}-\mathbf{ee}\right]\right.$$
$$\left.+(\mathbf{I}+\mathbf{ee}).\int_{\xi-\chi_1}^{\xi+\chi_1} d\eta\,\frac{\mathbf{N}}{|\eta-\xi|}\right\}(1+O(\epsilon^n)). \tag{A 12}$$

Equations (A 2), (A 6), (A 7) and (A 12) together show that (2.8) is a valid consequence of (2.7), provided that the previously mentioned end effects can be neglected, that assumption (A 1) is valid and that we can choose χ_1 and χ_2, and sensibly restrict $|\mathbf{R}_\xi|$ so that the inequalities (A 3), (A 5), (A 8), (A 9) and (A 10) are satisfied. The optimum values (in the sense that $|\mathbf{R}_\xi|$ achieves the widest range of values) of χ_1 and χ_2 to satisfy the inequalities are

$$\chi_2 = \alpha^\nu \quad \text{for} \quad 0 < \nu < 1, \quad \chi_1 = \alpha\epsilon^{-n}, \tag{A 13}$$

whence

$$|\mathbf{R}_\xi| = O(\alpha^{-1}\epsilon^{2n}). \tag{A 14}$$

This indicates that the theory is restricted to particles whose surface slopes are of $O(\epsilon^{2n})$, at least for the accuracy required.

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